MODIFIED RAYLEIGH CONJECTURE METHOD FOR MULTIDIMENSIONAL OBSTACLE SCATTERING PROBLEMS

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Abstract. The Rayleigh conjecture on the representation of the scattered field in the exterior of an obstacle D is widely used in applications. However this conjecture is false for some obstacles. AGR introduced the Modified Rayleigh Conjecture (MRC), and in this paper we present successful numerical algorithms based on the MRC for various 2D and 3D obstacle scattering problems. The 3D obstacles include a cube and an ellipsoid. The MRC method is easy to implement for both simple and complex geometries. It is shown to be a viable alternative for other obstacle scattering methods.

Key words and phrases: obstacle scattering, modified Rayleigh conjecture, numerical solution of obstacle scattering problem.

1. Introduction.

In this paper we present a novel numerical method for Direct Obstacle Scattering Problems based on the Modified Rayleigh Conjecture (MRC). The basic theoretical foundation of the method was developed in [7]. The MRC has the appeal of an easy implementation for obstacles of complicated geometry, e.g. having edges and corners. In our numerical experiments the method has shown itself to be a competitive alternative to the BIEM (boundary integral equations method), see [8]. Also, unlike the BIEM, one can apply the algorithm to different obstacles with very little additional effort.

However, a straightforward numerical implementation of the MRC, may be inefficient or fail. Nevertheless, one can make the numerical implementation of the method based on MRC work by introducing some new ideas within the same theoretical framework. In this paper we describe such an idea (Random Multi-point MRC) that made possible, for the first time, to apply the method successfully to 3D obstacles including a sphere and a cube.

In our previous paper [8] we described another implementation of the MRC. That method (Multi-point MRC) could be used for 2D obstacles of a relatively simple geometry, but it failed for some 2D obstacles, and it was not successful for 3D problems because of the problem size limitations. In this paper, in addition to treating 3D problems, we review our earlier results and show that they can be significantly improved by using the new idea for the implementaion of the MRC algorithm.

We formulate the obstacle scattering problem in a 3D setting with the Dirichlet boundary condition, but the method discussed can also be used for the Neumann and Robin boundary conditions.

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Consider a bounded domain $D \subset \mathbb{R}^3$, with a boundary S which is assumed to be Lipschitz continuous. Denote the exterior domain by $D' = \mathbb{R}^3 \backslash D$. Let $\alpha, \alpha' \in S^2$ be unit vectors, where S^2 is the unit sphere in $\mathbb{R}^3$.

The acoustic wave scattering problem by a soft obstacle D consists in finding the (unique) solution to the problem (1.1)-(1.2):

$$\left(\nabla^2 + k^2\right) u = 0 \text{ in } D', \quad u = 0 \text{ on } S, \tag{1.1}$$

$$u = u_0 + A(\alpha', \alpha)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \alpha' := \frac{x}{r}. \tag{1.2}$$

Here $u_0 := e^{ik\alpha\cdot x}$ is the incident field, $v := u - u_0$ is the scattered field, $A(\alpha', \alpha)$ is called the scattering amplitude, its k-dependence is not shown, $k > 0$ is the wavenumber. Denote

$$A_\ell(\alpha) := \int_{S^2} A(\alpha', \alpha)\overline{Y_\ell(\alpha')} d\alpha', \tag{1.3}$$

where $Y_\ell(\alpha)$ are the orthonormal spherical harmonics, $Y_\ell = Y_{\ell m}, -\ell \leq m \leq \ell$.

Informally, the Random Multi-point MRC algorithm can be described as follows.

Fix a $J > 0$. Let $x_j, j = 1, 2, ..., J$ be a batch of points randomly chosen inside the obstacle D. For $x \in D'$, let

$$\alpha' = \frac{x - x_j}{|x - x_j|}, \quad \psi_\ell(x, x_j) = Y_\ell(\alpha')h_\ell(k|x - x_j|), \tag{1.4}$$

where $h_\ell(r)$ are the spherical Hankel functions, normalized so that $h_\ell(r) \sim \frac{e^{ikr}}{r}$ as $r \to +\infty$.

Let $g(s) = u_0(s),\ s \in S$, and minimize the discrepancy

$$\Phi(\mathbf{c}) = \|g(s) + \sum_{j=1}^{J}\sum_{\ell=0}^{L} c_{\ell,j}\psi_\ell(s, x_j)\|_{L^2(S)}, \tag{1.5}$$

over $\mathbf{c} \in \mathbb{C}^N$, where $\mathbf{c} = \{c_{\ell,j}\}$. That is, the total field $u(s) = g(s) + v(s)$ is desired to be as close to zero as possible at the boundary S, to satisfy the required condition for soft scattering. If the resulting residual $r^{min} = \min \Phi$ is smaller than the prescribed tolerance ϵ, then the procedure is finished, and the sought scattered field is

$$v_\epsilon(x) = \sum_{j=1}^{J}\sum_{\ell=0}^{L} c_{\ell,j}\psi_\ell(x, x_j), \quad x \in D',$$

(see Lemma 2.2 below, and the Remark following it).

If, on the other hand, the residual $r^{min} > \epsilon$, then we continue by trying to improve on the already obtained fit in (1.5). Adjust the field on the boundary by letting $g(s) := g(s) + v_\epsilon(s),\ s \in S$. Create another batch of J points randomly chosen in the interior of D, and minimize (1.5) with this new $g(s)$. Continue with the iterations until the required tolerance ϵ on the boundary S is attained. In each iteration accumulate new interior points x_j and the corresponding best fit coefficients $c_{\ell,j}$. After the desired tolerance is reached, the sought scattered field v_ϵ is computed anywhere in D'.

Here is the precise description of the algorithm.

Random Multi-point MRC.

For $x_j \in D$, and $\ell \geq 0$ functions $\psi_\ell(x, x_j)$ are defined as in (1.4).

(1) **Initialization.** Fix $\epsilon > 0,\ L \geq 0,\ J > 0,\ N_{max} > 0$. Let $n = 0$, and $g(s) = u_0(s),\ s \in S$.
(2) **Iteration.**
   (a) Let $n := n + 1$. Randomly choose J points $x_j^{(n)} \in D,\ j = 1, 2, \ldots, J$.
   (b) Minimize
   $$\Phi(\mathbf{c}) = \|g(s) + \sum_{j=1}^{J} \sum_{\ell=0}^{L} c_{\ell,j} \psi_\ell(s, x_j^{(n)})\|_{L^2(S)}$$
   over $\mathbf{c} \in \mathbb{C}^N$, where $\mathbf{c} = \{c_{\ell,j}\}$.
   Let the minimum of Φ be attained at $\mathbf{c}^{(n)} = \{c_{\ell,j}^{(n)})\},\ j = 1, 2, \ldots, J$, and the minimal value of Φ be r^{min}.
(3) **Stopping criterion.**
   (a) If $r^{min} \leq \epsilon$, then stop. Compute the approximate scattered field anywhere in D' by
   $$v_\epsilon(x) := \sum_{k=1}^{n} \sum_{j=1}^{J} \sum_{\ell=0}^{L} c_{\ell,j}^{(k)} \psi_\ell(x, x_j^{(k)}), \quad x \in D'. \tag{1.6}$$
   (b) If $r^{min} > \epsilon$, and $n \neq N_{max}$, let
   $$g(s) := g(s) + \sum_{j=1}^{J} \sum_{\ell=0}^{L} c_{\ell,j}^{(n)} \psi_\ell(s, x_j^{(n)}), \quad x \in S$$
   and repeat the iterative step (2).
   (c) If $r^{min} > \epsilon$, and $n = N_{max}$, then the procedure failed.

2. Direct scattering problems and the Rayleigh conjecture.

Let a ball $B_R := \{x : |x| \leq R\}$ contain the obstacle D. In the region $r > R$ the solution to (1.1)-(1.2) is:

$$u(x, \alpha) = e^{ik\alpha\cdot x} + \sum_{\ell=0}^{\infty} A_\ell(\alpha)\psi_\ell, \quad \psi_\ell := Y_\ell(\alpha')h_\ell(kr), \quad r > R, \quad \alpha' = \frac{x}{r}, \tag{2.1}$$

where the sum includes the summation with respect to m, $-\ell \leq m \leq \ell$, and $A_\ell(\alpha)$ are defined in (1.3).

The Rayleigh conjecture (RC) is: the series (2.1) converges up to the boundary S (originally RC dealt with periodic structures, gratings). This conjecture is false for many obstacles, but is true for some ([1, 3, 9]). For example, if $n = 2$ and D is an ellipse, then the series analogous to (2.1) converges in the region $r > a$, where $2a$ is the distance between the foci of the ellipse [1]. In the engineering literature there are numerical algorithms, based on the Rayleigh conjecture. Our aim is to give a formulation of a *Modified Rayleigh Conjecture* (MRC) which holds for any Lipschitz obstacle and can be used in numerical solution of the direct and inverse scattering problems. We discuss the Dirichlet condition but a similar argument is applicable to the Neumann boundary condition, corresponding to acoustically hard obstacles.

Fix $\epsilon > 0$, an arbitrary small number.

Lemma 2.1. *There exist $L = L(\epsilon)$ and $c_\ell = c_\ell(\epsilon)$ such that*

$$||u_0 + \sum_{\ell=0}^{L(\epsilon)} c_\ell(\epsilon)\psi_\ell||_{L^2(S)} \le \epsilon. \tag{2.2}$$

If (2.2) and the boundary condition (1.1) hold, then

$$||v_\epsilon - v||_{L^2(S)} \le \epsilon, \quad v_\epsilon := \sum_{\ell=0}^{L(\epsilon)} c_\ell(\epsilon)\psi_\ell, \tag{2.3}$$

where v is the scattered field defined below formula (1.2).

Lemma 2.2. *If (2.3) holds then*

$$|||v_\epsilon - v||| = O(\epsilon)\,, \quad \epsilon \to 0, \tag{2.4}$$

where $|||\cdot||| := ||\cdot||_{H^m_{loc}(D')} + ||\cdot||_{L^2(D';(1+|x|)^{-\gamma})}$, $\gamma > 1$, $m > 0$ is an arbitrary integer, H^m is the Sobolev space, v is the scattered field, v_ϵ is defined in (2.3), and both v_ϵ and v in (2.4) are functions defined in D'.

In particular, (2.4) implies

$$||v_\epsilon - v||_{L^2(S_R)} = O(\epsilon)\,, \quad \epsilon \to 0, \tag{2.5}$$

where S_R is the sphere centered at the origin with radius R.

Remark. The proof of Lemma 2.2 in [7] shows that estimate (2.4) for the scattered field v in the exterior D' of the obstacle D follows from the boundary estimate

$$||v_\epsilon - v||_{L^2(S)} \le \epsilon$$

for any outgoing solution v_ϵ of $\left(\nabla^2 + k^2\right) v_\epsilon = 0$ in D', regardless of how such solution v_ϵ was constructed.

Lemma 2.3. *One has:*

$$c_\ell(\epsilon) \to A_\ell(\alpha)\,, \quad \forall \ell, \quad \epsilon \to 0. \tag{2.6}$$

The modified Rayleigh conjecture (MRC) is formulated as a theorem, which follows from the above three lemmas:

Theorem 2.4. *For an arbitrary small $\epsilon > 0$ there exist $L(\epsilon)$ and $c_\ell(\epsilon)$, $0 \le \ell \le L(\epsilon)$, such that (2.2), (2.4) and (2.6) hold.*

See [7] for a proof of the above statements.

The difference between RC and MRC is: (2.3) does not hold if one replaces v_ϵ by $\sum_{\ell=0}^{L} A_\ell(\alpha)\psi_\ell$, and lets $L \to \infty$ (instead of letting $\epsilon \to 0$). Indeed, the series $\sum_{\ell=0}^{\infty} A_\ell(\alpha)\psi_\ell$ diverges at some points of the boundary for many obstacles. Note also that the coefficients in (2.3) depend on ϵ, so (2.3) is *not* a partial sum of a series.

For the Neumann boundary condition one minimizes

$$\left\| \frac{\partial[u_0 + \sum_{\ell=0}^{L} c_\ell\psi_\ell]}{\partial N} \right\|_{L^2(S)}$$

with respect to c_ℓ. Analogs of Lemmas 2.1-2.3 are valid and their proofs are essentially the same.

See [10] for an extension of these results to scattering by periodic structures.

3. Numerical Experiments.

In this section we describe numerical results obtained by the Random Multi-point MRC method for 2D and 3D obstacles. We also compare the 2D results to the ones obtained by our earlier method introduced in [8]. The method that we used previously can be described as a Multi-point MRC. It was not an iterative method, and its applicability was limited by the problem size, i.e. the number J of the interior points x_j. It required extensive computational resources (run time and memory) needed for the minimization part of the algorithm even for a moderate number J.

The Multi-point MRC method is just the first iteration of the Random method. It can be run efficiently with a relatively modest J, where J is kept constant across iterations. Also, in the Multi-point MRC method the interior points $x_j,\ j = 1, 2, ..., J$, were chosen deterministically by an *ad hoc* method according to the geometry of the obstacle D.

The Random Multi-point MRC eliminates the need for this special procedure. See the next section for an additional discussion of this issue.

As we mentioned previously, [8] contains a favorable comparison of the Multi-point MRC method with the Boundary Integral Equation Method, despite the fact that the numerical implementation of the MRC method in [8] is considerably less efficient than the one presented in this paper.

A numerical implementation of the Random Multi-point MRC method follows the same outline as for the Multi-point MRC, which was described in [8]. Of course, in a 2D case, instead of (1.4) one has

$$\psi_l(x, x_j) = H_l^{(1)}(k|x - x_j|)e^{il\theta_j},$$

where $(x - x_j)/|x - x_j| = e^{i\theta_j}$.

For a numerical implementation choose M nodes $\{t_m\}$ on the surface S of the obstacle D. After the interior points $x_j,\ j = 1, 2, ..., J$ are chosen, form N vectors

$$\mathbf{a}^{(n)} = \{\psi_l(t_m, x_j)\}_{m=1}^{M},$$

$n = 1, 2, \ldots, N$ of length M. Note that $N = (2L + 1)J$ for a 2D case, and $N = (L + 1)^2 J$ for a 3D case. It is convenient to normalize the norm in $\mathbb{R}^M$ by

$$\|\mathbf{b}\|^2 = \frac{1}{M}\sum_{m=1}^{M} |b_m|^2, \quad \mathbf{b} = (b_1, b_2, ..., b_M).$$

Then $\|u_0\| = 1$.

Now let $\mathbf{b} = \{g(t_m)\}_{m=1}^{M}$, in the Random Multi-point MRC (see section 1), and minimize

$$\Phi(\mathbf{c}) = \|\mathbf{b} + A\mathbf{c}\|, \tag{3.1}$$

for $\mathbf{c} \in \mathbb{C}^N$, where A is the matrix containing vectors $\mathbf{a}^{(n)},\ n = 1, 2, \ldots, N$ as its columns.

We used the Singular Value Decomposition (SVD) method (see e.g. [4]) to minimize (3.1). Small singular values $s_n < w_{min}$ of the matrix A are used to identify and delete linearly dependent or almost linearly dependent combinations of vectors $\mathbf{a}^{(n)}$. This spectral cut-off makes the minimization process stable, see the details in [8].

Table 1. Normalized residuals attained in the numerical experiments for 2D obstacles, $\|\mathbf{u_0}\| = 1$.

Experiment	J	k	α	r_{old}^{min}	r^{min}
I	4	1.0	$(1.0, 0.0)$	0.000201	0.0001
	4	1.0	$(0.0, 1.0)$	0.000357	0.0001
	4	5.0	$(1.0, 0.0)$	0.001309	0.0001
	4	5.0	$(0.0, 1.0)$	0.007228	0.0001
II	16	1.0	$(1.0, 0.0)$	0.003555	0.0001
	16	1.0	$(0.0, 1.0)$	0.002169	0.0001
	16	5.0	$(1.0, 0.0)$	0.009673	0.0001
	16	5.0	$(0.0, 1.0)$	0.007291	0.0001
III	16	1.0	$(1.0, 0.0)$	0.008281	0.0001
	16	1.0	$(0.0, 1.0)$	0.007523	0.0001
	16	5.0	$(1.0, 0.0)$	0.021571	0.0001
	16	5.0	$(0.0, 1.0)$	0.024360	0.0001
IV	32	1.0	$(1.0, 0.0)$	0.006610	0.0001
	32	1.0	$(0.0, 1.0)$	0.006785	0.0001
	32	5.0	$(1.0, 0.0)$	0.034027	0.0001
	32	5.0	$(0.0, 1.0)$	0.040129	0.0001

Let r^{min} be the residual, i.e. the minimal value of $\Phi(\mathbf{c})$ attained after N_{max} iterations of the Random Multi-point MRC method (or when it is stopped). For a comparison, let r_{old}^{min} be the residual obtained in [8] by an earlier method.

We conducted 2D numerical experiments for four obstacles: two ellipses of different eccentricity, a kite, and a triangle. The M=720 nodes t_m were uniformly distributed on the interval $[0, 2\pi]$, used to parametrize the boundary S. Each case was tested for wave numbers $k = 1.0$ and $k = 5.0$. Each obstacle was subjected to incident waves corresponding to $\alpha = (1.0, 0.0)$ and $\alpha = (0.0, 1.0)$.

The results for the Random Multi-point MRC with $J = 1$ are shown in Table 1, in the last column r^{min}. In every experiment the target residual $\epsilon = 0.0001$ was obtained in under 6000 iterations, in about 2 minutes run time on a 2.8 MHz PC.

In [8], we conducted numerical experiments for the same four 2D obstacles by a Multi-point MRC, as described in the beginning of this section. The interior points x_j were chosen differently in each experiment. Their choice is indicated in the description of each 2D experiment. The column J shows the number of these interior points. Values $L = 5$ and $M = 720$ were used in all the experiments. These results are shown in Table 1, column r_{old}^{min}.

Thus, the Random Multi-point MRC method achieved a significant improvement over the earlier Multi-point MRC.

Experiment 2D-I. The boundary S is an ellipse described by

$$\mathbf{r}(t) = (2.0\cos t,\ \sin t), \quad 0 \le t < 2\pi\,. \tag{3.2}$$

The Multi-point MRC used $J = 4$ interior points $x_j = 0.7\mathbf{r}(\frac{\pi(j-1)}{2}),\ j = 1, \ldots, 4$. The run time was 2 seconds.

Experiment 2D-II. The kite-shaped boundary S (see [2], Section 3.5) is described by

$$\mathbf{r}(t) = (-0.65 + \cos t + 0.65\cos 2t,\ 1.5\sin t), \quad 0 \le t < 2\pi\,. \tag{3.3}$$

TABLE 2. Normalized residuals attained in the numerical experiments for 3D obstacles, $\|\mathbf{u_0}\| = 1$.

Experiment	k	α_i	r^{min}	N_{iter}	run time
I	1.0		0.0002	1	1 sec
	5.0		0.001	700	7 min
II	1.0	(1)	0.001	800	16 min
	1.0	(2)	0.001	200	4 min
	5.0	(1)	0.0035	2000	40 min
	5.0	(2)	0.002	2000	40 min
III	1.0	(1)	0.001	3600	37 min
	1.0	(2)	0.001	3000	31 min
	5.0	(1)	0.0026	5000	53 min
	5.0	(2)	0.001	5000	53 min

The Multi-point MRC used $J = 16$ interior points $x_j = 0.9\mathbf{r}(\frac{\pi(j-1)}{8}),\ j = 1, \dots, 16$. The run time was 33 seconds.

Experiment 2D-III. The boundary S is the triangle with vertices at $(-1.0, 0.0)$ and $(1.0, \pm 1.0)$. The Multi-point MRC used the interior points $x_j = 0.9\mathbf{r}(\frac{\pi(j-1)}{8})$, $j = 1, \dots, 16$. The run time was about 30 seconds.

Experiment 2D-IV. The boundary S is an ellipse described by

$$\mathbf{r}(t) = (0.1\cos t,\ \sin t), \quad 0 \le t < 2\pi\,. \tag{3.4}$$

The Multi-point MRC used $J = 32$ interior points $x_j = 0.95\mathbf{r}(\frac{\pi(j-1)}{16}),\ j = 1, \dots, 32$. The run time was about 140 seconds.

The 3D numerical experiments were conducted for 3 obstacles: a sphere, a cube, and an ellipsoid. We used the Random Multi-point MRC with $L = 0$, $w_{min} = 10^{-12}$, and $J = 80$. The number M of the points on the boundary S is indicated in the description of the obstacles. The scattered field for each obstacle was computed for two incoming directions $\alpha_i = (\theta, \phi),\ i = 1, 2$, where ϕ was the polar angle. The first unit vector α_1 is denoted by (1) in Table 2, $\alpha_1 = (0.0, \pi/2)$. The second one is denoted by (2), $\alpha_2 = (\pi/2, \pi/4)$. A typical number of iterations N_{iter} and the run time on a 2.8 MHz PC are also shown in Table 2. For example, in experiment I with $k = 5.0$ it took about 700 iterations of the Random Multi-point MRC method to achieve the target residual $r^{min} = 0.001$ in 7 minutes.

Experiment 3D-I. The boundary S is the sphere of radius 1, with $M = 450$.

Experiment 3D-II. The boundary S is the surface of the cube $[-1, 1]^3$ with $M = 1350$.

Experiment 3D-III. The boundary S is the surface of the ellipsoid $x^2/16 + y^2 + z^2 = 1$ with $M = 450$.

In the last experiment the run time could be reduced by taking a smaller value for J. For example, the choice of $J = 8$ reduced the running time to about 6-10 minutes.

Numerical experiments show that the minimization results depend on the choice of such parameters as J, w_{min}, and L.

4. Discussion of the results.

Let D be an obstacle, S be its boundary, and u_0 be the incident field. It is proved in [7] that if v_ϵ is an outgoing solution of the Helmholtz equation in the exterior domain D' and u_0+v_ϵ approximates zero in $L^2(S)-$norm on the boundary S, then v_ϵ approximates the exact scattered field v in D', see Lemma 2.2 and the Remark after it. The Modified Rayleigh Conjecture approach to obstacle scattering problems is based on the following observation: the functions $\psi_\ell(x,z),\ \ z \in D$ and their linear combinations are outgoing solutions to the Helmholtz equation in the exterior domain. Therefore, one just needs to find a combination of such functions that gives the best fit to $-u_0$ on the boundary S. Then this combination approximates the scattered field everywhere in the exterior D' of the obstacle D and the error of this approximation is given in Theorem 2.4.

Various implementations of the MRC method provide different algorithms for the construction of such best fit. The original theory of MRC method given in [7] guarantees that one can use all $\psi_\ell(x,z)$ at $z=0$ to obtain the required fit. However, it is not always possible to obtain the desired accuracy numerically, because this would require a very high accuracy of computations due to the fact that the Hankel functions differ from each other by many orders of magnitude if ℓ is large. Thus, one wants to restrict the order L. We found numerically that $L=5$ is a reasonable value from the numerical point of view.

To keep L reasonably small, we suggested in [8] to use a batch of interior points $x_j \in D,\ j=1,2\ldots,J$, and the associated source functions $\psi_\ell(x,x_j)$ to find the best fit (Multi-point MRC). The results are presented in [8]. They improve with the increase in the number J of the interior points. However, for $J>30$ the problem size becomes too large for our computer system to handle it efficiently, thus limiting the applicability of the Multi-point MRC. Still, if one can achieve a satisfactory fit with this method (2D or 3D), then it is more efficient than BIEM. Another issue in the Multi-point MRC is the placement of the interior points x_j. In [8] we used an *ad hoc* procedure, by spreading the points just behind the boundary S of the obstacle.

In the present paper we made a further advance in the MRC algorithm. Our Random Multi-point MRC is an iterative method. Thus it allows to keep the number J of the interior point relatively small, but still achieve a good boundary fit. Undoubtedly, there are many strategies as to how to place the interior points x_j. Clearly, they should be chosen distinctly in subsequent iterations. Numerical experiments showed that restricting these points to a subset of the obstacle D did not produce satisfactory relults. We tried to place the points in the interior of D in a random fashion, to assure that the entire interior is utilized. The results of this algorithm are presented in this paper. They show a significant improvement in the obtained fit as compared with our earlier method. Also, we were able to obtain accurate solutions for 3D problems. Clearly, the random choice of the points is not essential to the algorithm. We think that a suitable deterministic choice of points could be as successful and are working on finding such a choice.

It may happen that subsequent iterations bring only a negligible improvement to the already obtained minimization values of Φ in the method we proposed in this paper, and we do not have a theoretically justified way around this difficulty.

We are trying to find an optimal placement for the interior points and we hope to report the results of this research upon its completion.

5. Conclusions.

For a 2D, or 3D obstacle, Rayleigh conjectured that the acoustic field u in the exterior of the obstacle is given by

$$u(x,\alpha) = e^{ik\alpha\cdot x} + \sum_{\ell=0}^{\infty} A_\ell(\alpha)\psi_\ell, \quad \psi_\ell := Y_\ell(\alpha')h_\ell(kr), \quad \alpha' = \frac{x}{r}. \tag{5.1}$$

While this conjecture (RC) is false for many obstacles, it has been modified to obtain a representation for the solution of (1.1)-(1.2) and to obtain its error. This representation (Theorem 2.4) is called the Modified Rayleigh Conjecture (MRC). In fact, it is not a conjecture, but a theorem.

We propose here an implementation of the MRC method which gives an efficient approach to solving obstacle scattering problems in 3D problems with complicated geometries. Our implementation of the MRC method worked in the cases considered more efficiently than the BIEM method.

The implementation of the MRC method presented in this paper, the Random Multi-point MRC, has been successfully applied to various 2D and 3D obstacle scattering problems. This algorithm is a significant improvement over the earlier implementation of MRC method, given in [8]. The improvement is achieved by allowing the required minimizations to be done iteratively. However, even the earlier implementation (see [8]) compared favorably to the BIEM.

The Random Multi-point MRC has an additional attractive feature: it can easily treat obstacles with complicated geometry (e.g. edges and corners). Unlike the BIEM, the Random Multi-point MRC can be relatively easily used for solving obstacle scattering problems with complicated geometries and rough boundaries.

Further research on MRC algorithms is under way. The authors hope that a numerical implementation of the MRC method will yield a more efficient and economical algorithm for solving obstacle scattering problems than the currently used methods.

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